Loewner chains generated by slit domains

Consider a very special case of slit domains.

Let $\gamma(t) : [0,\infty] \to \mathbb{D}$, $\gamma(0) \in \partial\mathbb{D}$, $\gamma(\infty) = 0$, $\gamma(0,\infty) \subset \mathbb{D}\setminus\{0\}$, simple (or self-touching). Normalize so that $f_t : \mathbb{D} \to \Omega_t$ has $|f_t'(0)| = e^{-t}$.

Then $\Omega_t :=$ component of 0 of $\mathbb{D}\setminus\gamma[0,t]$ normalized L.C.

f_t extends continuously to $\overline{\mathbb{D}}$ (by Carathéodory!) Let $\lambda(t) = f_t^{-1}(\gamma(t))$.
(Ω_t for self-touching)

As before: $g_t := f_t^{-1}$.

$\varphi_{s,t} = f_s^{-1} \circ f_t$, $s \le t$

Theorem. $\frac{d}{dt} g_t(z) = g_t(z) \frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}$ $\quad \frac{\partial}{\partial t} f_t(z) = -z f_t'(z) \frac{\lambda(t)+z}{\lambda(t)-z}$

Remark $\boxed{\frac{\partial f_t(z)}{\partial t} = -f_t'(z)\, z \int \frac{\zeta+z}{\zeta-z}\, d\mu_t}$ – General Löwner evolution

The theorem means that $\mu_t = \delta_{\lambda(t)}$, $\lambda(t) \in C(\mathbb{R}_+)$

$\lambda(t)$ is called the driving function of the Löwner chain (f_t).

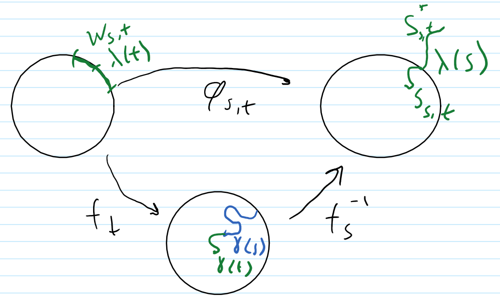
$\frac{\partial}{\partial t}$ now exists for all t.

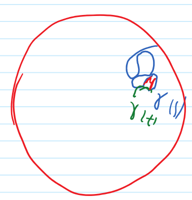
Proof

Reminder:
$p_{s,t}(z) = p(z,s,t) := \frac{1+e^{(s-t)}}{1-e^{(s-t)}} \frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)} \in \mathcal{P}$. $s \le t$.

So $p_{s,t}(z) = \int \frac{\zeta+z}{\zeta-z} d\mu_{s,t}(z)$, $\operatorname{supp} \mu_{s,t}(z) = \operatorname{clos}\{\zeta \in \mathbb{D} : \lim_{r\to 1^-} |\varphi_{s,t}(r\zeta)| \ne 1\}$.

Here it is $f_t^{-1} \circ f_s(\gamma) = f_t^{-1}(\gamma[s,t]) = g_t(\gamma[s,t]) =: W_{s,t}$





Let $S_{s,t} := g_s(\gamma[s,t]) = \varphi_{s,t}(W(s,t))$

Use Schwarz reflection to expand $\varphi_{s,t}$ to map $\hat{\mathbb{C}} \setminus W(s,t) \to \hat{\mathbb{C}} \setminus (S_{s,t} \cup S_{s,t}^*)$.

By Lavrentiev: $\lim_{t \downarrow s} \operatorname{diam} S_{s,t} = 0$ ($f_s(S_{s,t}) = \gamma[s,t]$, and as $t \downarrow s$, $\operatorname{diam} \gamma[s,t] \downarrow 0$).
$\lim_{s \uparrow t} \operatorname{diam} S_{s,t} = 0$. So when $t \downarrow s$, $S_{s,t}$ approaches $\lambda(s)$

Also by Lavrentiev: $\lim_{\substack{t \to s \\ s \to t}} \operatorname{diam} W_{s,t} = 0$.

Remark. To establish $\lim_{\substack{s \to t \\ t \to s}} \operatorname{diam} S_{s,t} = 0$ and $\lim_{\substack{s \to t \\ t \to s}} \operatorname{diam} W_{s,t} = 0$, we only used the fact that one can find a small crosscut σ in Ω_s separating $\Omega_s \setminus \Omega_t$ from 0.

The rest of the proof does not use γ at all and only uses this estimate.

Let us prove that $\varphi_{s,t}(z) \to z$ on $\mathbb{C}$ (where defined)

Consider $\psi_{s,t}(z) := \frac{\varphi_{s,t}(z)}{z}$ – analytic in $\mathbb{C} \setminus W_{s,t}$ (singularity at 0 is removable)

Bounded at ∞ as long as $s < t \leq T$:

$$\lim_{z \to \infty} \psi_{s,t}(z) = \lim_{z \to \infty} \frac{\varphi_{s,t}(z)}{z} = \lim_{z \to 0} \frac{z}{\overline{\varphi_{s,t}}(z)} = \frac{1}{\overline{\varphi}'_{s,t}(0)} = e^{t-s}$$

By Köbe $\frac{1}{4}$-Theorem: $\forall z \in S_{s,t}$: $|z| \geq \frac{e^{s-t}}{4}$; so $\forall z \in S^*_{s,t}$, $|z| \leq 4e^{t-s}$

So on $W_{s,t}$, $|\psi_{s,t}(z)| \leq 4e^{t-s}$. So, by maximum principle, $|\psi_{s,t}| \leq 4e^{t-s} \leq 4e^T$ as long as $s < t \leq T$. So $\{\psi_{s,t}\}_{s < t \leq T}$ – form a normal family.

Consider any subsequential limit $\psi := \lim_{s_n \uparrow t} \psi_{s_n,t}$. Then ψ is bounded by e^T and analytic in $\mathbb{C} \setminus \{\lambda(t)\}$ (since $\operatorname{diam} W_{s,t} \to 0$). So $\lambda(t)$ is removable. By Liouville, $\psi(z) \equiv \text{const} = \psi(0) = 1$.

So, by normality, $\lim_{s \uparrow t} \psi_{s,t}(z) = 1 \Rightarrow \lim_{s \uparrow t} \varphi_{s,t}(z) = z$.

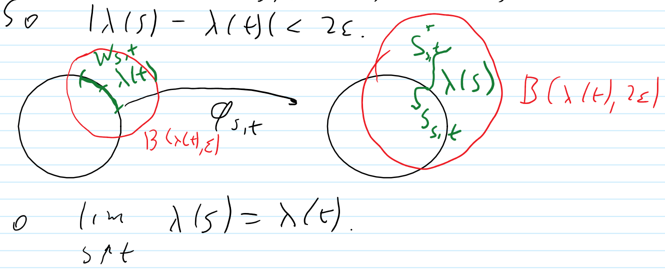
Take s close enough to t so that $W_{s,t} \subset B(\lambda(t), \varepsilon)$.

Then, for s close to t, $|\varphi_{s,t}(z) - z| < \varepsilon$ on $\{|z - \lambda(t)| = \varepsilon\}$.

So $\varphi_{s,t}(\{|z - \lambda(t)| < \varepsilon\}) \subset B(\lambda(t), 2\varepsilon)$.

But $\lambda(s) \in \varphi_{s,t}(W_{s,t}) = S_{s,t} \subset B(\lambda(t), 2\varepsilon)$.

So $|\lambda(s) - \lambda(t)| < 2\varepsilon$.



So $\lim_{s \uparrow t} \lambda(s) = \lambda(t)$.

To establish $\lim_{t \downarrow s} \lambda(t) = \lambda(s)$, apply the same argument to $\varphi^{-1}_{s,t}(z)$.

Reminder:

$$\frac{f_t(z)-f_s(z)}{t-s} = \underbrace{\frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z}} \cdot \underbrace{\int \frac{\zeta - z}{\zeta + z}\, d\mu_{s,t}(\zeta)} \underbrace{\frac{(e^{s-t}-1)(z+\varphi_{s,t}(z))}{(1+e^{s-t})(t-s)}}$$

$\mu_{s,t}$ is supported on $W_{s,t} := \{\zeta \in \mathbb{T} : \varphi_{s,t}(\zeta) \notin \mathbb{T}\}$.

As $s \to t$, $\mu_{s,t} \to \delta_{\lambda(t)}$ (since it is the only probability measure supported on $\{\lambda(t)\}$).

Thus $\lim\limits_{s \to t} \int \frac{\zeta - z}{\zeta + z}\, d\mu_{s,t}(\zeta) = \frac{\lambda(t) - z}{\lambda(t) + z}$ $\forall\, t, z$.

Also, as before,

$$\lim_{s \to t} \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{\varphi_{s,t}(z) - z} \to f_t'(z),$$

$$\lim_{s \to t} \frac{e^{(s-t)}-1}{1+e^{s-t}} \frac{(z+\varphi_{s,t}(z))}{t-s} = -z.$$

So $\frac{\partial f_t}{\partial t}$ exists for all z and t, and equal to

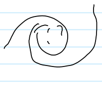
$-z f_t'(z) \frac{\lambda(t)+z}{\lambda(t)-z}$ $\blacksquare$

Christian Pommerenke

What can be generated by continuous functions? Unfortunately, not only curves.

Thm. (Pommerenke) The L.C. (Ω_t) has a continuous driving function $\lambda(t)$ iff $\forall T > 0$, $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall t \le T$ $\exists$ crosscut γ in Ω_t separating 0 from $\Omega_t \setminus \Omega_{t+\delta}$, $\operatorname{diam} \gamma < \varepsilon$.

Example (bad) Spiral.



Pf. The existence and continuity of $\lambda(t)$: exactly as in previous thm, by the remark.

Other direction

Fix $\eta > 0$, $\delta < \frac{\eta^2}{16}$: $|t - s| \le \delta \Rightarrow |\lambda(t) - \lambda(s)| < \frac{\eta}{4}$. $s \le t \le T$

Claim 1. $|z| < 1$, $|z - \lambda(s)| > \eta$. Then $u(t) := |\lambda(s) - \varphi_{s,t}(z)| > \eta/2$ and $|z - \varphi_{s,t}(z)| < \frac{\eta}{2}$ for $s \le t \le s + \delta$.

Meaning: If z was far away from $\lambda(s)$, it stays far away for time δ.

Pf. Let $t_1 = \inf\{t : u(t) = \eta/2\}$. For $s \le t \le t_1$, $u(t) \ge \frac{\eta}{2}$. So $|\lambda(t) - \varphi_{s,t}(z)| \ge |\lambda(s) - \varphi_{s,t}(z)| - |\lambda(t) - \lambda(s)| \ge \frac{\eta}{4}$ (if $t_1 - s < \delta$)

so $\frac{d}{dt} u(t) = \frac{\partial}{\partial t} |\lambda(s) - \varphi_{s,t}(z)| \ge -|\varphi_{s,t}(z)| \left| \frac{\lambda(t) + \varphi_{s,t}(z)}{\lambda(t) - \varphi_{s,t}(z)} \right| \ge -\frac{8}{\eta}$.

$u(t_1) - u(s) \ge -\frac{8}{\eta}(t_1 - s) \ge -\frac{8\delta}{\eta} > -\frac{\eta}{2}$ — contradiction with $t_1 < s + \delta$ ∎

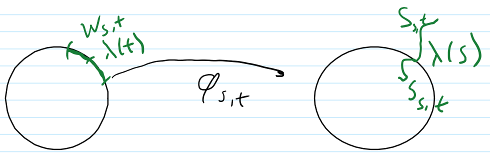
Claim 2. Let $|z - \lambda(s)| > \eta$. Then $|\varphi_{s,t}(z)| > |z|^2$ when $s \le t \le s + \delta$.

Pf. Let $t_2 = \inf\{t : |\varphi_{s,t}(z)| = |z|^2\}$.

Then

$$\partial_t \log|\varphi_{s,t}(z)| = \operatorname{Re}\left|\frac{1}{\varphi}\frac{\partial\varphi}{\partial t}\right| = -\operatorname{Re}\left(\frac{\lambda(t) + \varphi}{\lambda(t) - \varphi}\right) = \frac{|\varphi|^2 - 1}{|\lambda(t) - \varphi|^2}.$$

As before can see that $\log\varphi$ cannot go from $\log|z|$ to $2\log|z|$ in time δ ∎



$S_{s,t} = \mathbb{D} \setminus \varphi_{s,t}(\mathbb{D}) = f_s^{-1}(\Omega_s \setminus \Omega_t)$, as before

By Claims 1 and 2, if $s \le t \le s + \delta$, then $S_{s,t} \subset \{z \in \mathbb{D} : |z - \lambda(s)| < 2\eta\}$.

Indeed, $w \in S_{s,t} \Rightarrow w = \varphi_{s,t}(z)$, $|w| < |z|^2$, so, by Claim 2, $|z - \lambda(s)| \le \eta$.

(strictly speaking, $w_n = \varphi_{s,t}(z_n) \to w \Rightarrow |w_n| < |z_n|^2 \Rightarrow |z_n - \lambda(s)| \le \eta$
$\Rightarrow |z - \lambda(s)| \le \eta$).

So, if $D_\eta := \{z : |z - \lambda(s)| > \eta\}$, we get : $\varphi_{s,t}(D_\eta) \cap S_{s,t} = \emptyset$.

On the other hand:

$\varphi_{s,t}(D_\eta) \supset \{w : |w - \lambda(t)| \ge 2\eta\}$, since $|\lambda(s) - \lambda(t)| < \eta/4$ and Claim 1.

$$\left(|w - \lambda(t)| > 2\eta \underset{|\lambda(s) - \lambda(t)| < \frac{\eta}{4}}{\Rightarrow} \underset{|\varphi_{s,t}(z) - \lambda(s)|}{\overset{\shortparallel}{|w - \lambda(s)|}} > \frac{7}{4}\eta \underset{\text{Claim 1}}{\Rightarrow} |z - \lambda(s)| > \eta \right)$$

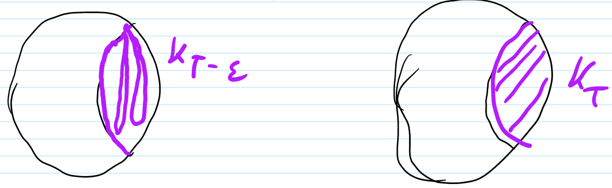
So, by Wolf Lemma, we can find a short crosscut in $f_s(\mathbb{D}) = \Omega_s$ separating $\Omega_t \setminus \Omega_s = f_s(S_{s,t})$ from 0. ∎

Def. L.C. is generated by a right-continuous curve γ if Ω_t = component of 0 of $\mathbb{D} \setminus \gamma[0,t]$.

Thm. TFAE for L.C. with a continuous driving function $\lambda(t)$:

1) L.C. is generated by a curve,
2) K_t is locally connected $\forall t$.

Why we need right-continuity:



Not left continuous at T.

But if $\gamma(t) = \lim_{r \to 1-} f_t(r\lambda(t))$ is continuous $\Rightarrow$ generated by continuous curve – Don't even need to assume local connectivity.

Pf. 1) $\Rightarrow$ 2) – obvious.

2) $\Rightarrow$ 1) By local connectivity, $\exists \lim_{r\to 1-} f_t(r\lambda(t)) = \gamma(t)$, right continuous (by Pommerenke).

Enough to prove: $\partial\Omega_t \subset \overline{\gamma[0,t]} \cup \partial\mathbb{D}$. Ω_t – component of 0 of $\mathbb{D} \setminus \gamma[0,t]$

Observe: If $\sigma: [0,1]$ – semicrosscut in Ω_t (i.e. $\sigma: [0,1] \to \overline{\Omega}_t$, $\sigma([0,1)) \subset \Omega_t$, $\sigma(1) \in \partial\Omega_t$), and if $\sigma(1) \in K_t \setminus \bigcup_{s<t} K_s$ then $\sigma(1) = \lambda(t)$.

Indeed, $f_t^{-1}(\sigma)$ is a semi-crosscut in $\mathbb{D}$ which lands at a point of $\mathbb{T}$ which corresponds to a prime end in $K_t \setminus \bigcup_{s<t} K_s$.

By Pommerenke's Thm, there is only one such prime end, $\lambda(t)$, so $\sigma(1) = f_t(\lambda(t)) = \gamma(t)$.

– well-defined, local connectivity

Let $z \in \partial\Omega_t \setminus \partial\mathbb{D}$, $\varepsilon > 0$, and $t' = \sup\{s : K_s \cap B(z,\varepsilon) = \emptyset\}$. $t' \le t$.

Take $p \in \overline{B(z,\varepsilon)} \cap \Omega_{t'}$, $p' \in K_{t'} \cap \overline{B(z,\varepsilon)}$. And let p'' be the first point of segment from z to p' which is in $K_{t'}$. Then $[z\, p'')$ – semi-crosscut in $\Omega_{t'}$. So $p'' = \lambda(t')$, by previous remark.

So $|z - \lambda(t')| \le \varepsilon$. So $z \in \overline{\gamma(0,t)}$.

Joan Lind

The best deterministic condition so far:

Thm (Lind) 1) If $\lambda(t) \in \text{Höl}_{1/2}$, $|\lambda(t)-\lambda(s)| \le c|s-t|^{1/2}$, $c<4$, then K_t is generated by a curve.

2) $\exists\, \lambda(t) \in \text{Höl}_{1/2}$, $|\lambda(t)-\lambda(s)| \le 4|s-t|^{1/2}$, such that K_t is <u>not</u> generated by a curve.